An English Translation:

Mathematics for Dynamical Systems

6

Let $n > 1$ be an integer and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Consider the system of differential equations

$$\frac{dx}{dt} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

on $\mathbb{R}$. Let $x = \phi(t)$ be a bounded nonconstant solution to Eq. (1) on $\mathbb{R}$. The following equation is called the variational equation of Eq. (1) around the solution $x = \phi(t)$:

$$\frac{dy}{dt} = Df(\phi(t))y, \quad y \in \mathbb{R}^n. \tag{2}$$

Here $Df(x)$ is the Jacobian matrix of $f(x)$ and given by the $n \times n$ matrix

$$Df(x) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(x) & \cdots & \dfrac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(x) & \cdots & \dfrac{\partial f_n}{\partial x_n}(x) \end{pmatrix},$$

where $f_j(x)$ and x_j are the j-th elements of $f(x)$ and x, respectively, for $j = 1, 2, \ldots, n$. Answer the following questions.

(i) When there exist the limits $a_+ = \lim_{t \to +\infty} \phi(t)$ and $a_- = \lim_{t \to -\infty} \phi(t)$, show that $x = a_+$ and a_- are constant solutions to Eq. (1). In addition, show that the variational equation (2) has a bounded solution $y = \psi(t)$ on $\mathbb{R}$ such that $\lim_{t \to \pm\infty} \psi(t) = 0$.

(ii) Assume that there exists a C^1 function $v : \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$Dv(x)f(x) - Df(x)v(x) = 0.$$

Obtain two linearly independent solutions to the variational equation (2) when the two vectors $f(\phi(0))$ and $v(\phi(0))$ are linearly independent.

(iii) Assume that there exist $n-1$ C^1 functions $v_j : \mathbb{R}^n \to \mathbb{R}^n$ $(j = 1, 2, \ldots, n-1)$ satisfying

$$Dv_j(x)f(x) - Df(x)v_j(x) = 0 \quad (j = 1, 2, \ldots, n-1).$$

Obtain a general solution to the variational equation (2) when the n vectors $f(\phi(0))$ and $v_j(\phi(0))$ $(j = 1, 2, \ldots, n-1)$ are linearly independent.